Mathematical Fundamentals for Understanding Reactive Power and Power Factor in the Time Domain

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Abstract: The purpose of this paper is to provide the reader with the mathematical fundamentals needed to understand reactive power and power factor and to follow some of the latest research in the field. From personal experience, the author has found that many professionals working in the electric utility industry either have an inadequate understanding of power factor and reactive power or simply have difficulty explaining the concepts to other people. The author suspects that one of the principal reasons for this is that power factor and reactive power are often presented in terms of complex power (i.e., in the "frequency" domain) and in terms of the familiar power factor triangle without describing the basic mathematics and underlying circuit equations used to develop these concepts. Utility customers and regulatory staff members with a rudimentary technical background will often grasp the basic principles of reactive power if the subject is first introduced in the time domain before explaining the concepts in the frequency domain.

I. INTRODUCTION

After the 2003 blackout in the United States and Canada there was a renewed interest in reactive power and power factor improvement in the electric utility industry. The Federal Energy Regulatory Commission ("FERC") allowed independent generators to charge utilities and their customers for reactive power service simply because the generators were connected to the grid. On February 4, 2005, the FERC published a Staff Report titled *Principles for Efficient and Reliable Reactive Power Supply and Consumption* detailing problems and concerns with the pricing of reactive power services. Utilities around the country are implementing stricter power factor provisions in both wholesale power contracts and in retail electric tariffs. It is therefore important that engineers, rate analysts, and marketing professionals working for electric utilities have a strong enough understanding of these concepts to explain the need for power factor adjustments to customers and regulators.

Having worked in the electric utility industry for almost 30 years, I have found that many rate managers, marketing representatives, and even practicing electrical engineers have difficulty providing a basic description of reactive power and power factor. Power factor and reactive power are often described using the power factor triangle with "real power" (measured in kW) corresponding to the adjacent leg, with "reactive" or "imaginary" power (measured in kVar) corresponding to the opposite leg, and "apparent" power corresponding to the hypotenuse of a right triangle. Although this picture, which is based on representing AC power as a complex number using phasor representations of current and voltage, is easy to remember and enormously important for performing calculations, it provides little insight into the causes of reactive power and the need for implementing rate provisions to encourage customers with large inductive loads to improve their power factor.

Based on personal experience, customers and regulatory commission staffs seem to understand power factor better when it is first presented in the "time domain" by representing current, voltage and power as time dependent signals rather than fixed legs in a triangle or by using a folksy metaphor to evoke the "meaning" of these concepts. Describing reactive power as "foam on the beer" or explaining the Pythagorean relationship of complex power in terms of "a horse pulling a boat down a canal" tends to make reactive power more mysterious than it really is. Reactive power and power factor are better understood by delving into the underlying mathematics than trying to plumb the hidden meaning of a metaphor.

In this paper, reactive power and power factor will be described mathematically in the time domain based on the sinusoidal representation of AC current and voltage. But prior to providing a mathematical description, these concepts will be described briefly in non-mathematical terms; therefore, anyone who wants to pass over the mathematical discussion can skip the remainder of the paper. For those interested in working through the mathematics (which may be difficult for people without a background in engineering, mathematics, or physics), this paper will hopefully help readers develop a stronger understanding of reactive power and power factor and perhaps provide a structure for explaining these concepts to customers and regulators. Another purpose of the paper is to provide the basic mathematical tools which would allow technicians to keep abreast of some of the latest developments in the area of power factor control. One of the principal reasons that this paper was written was to provide interested readers a presentation of the subject matter that I would have liked to have seen when I began working as an applied mathematician in a utility rate department almost 30 years ago.

II. NON-MATHEMATICAL OVERVIEW

Power factor is a measure of the efficiency of the utilization of the current and voltage supplied to customers. Specifically, a customer's *power factor* is the relationship of the power that is used by the customer (or converted to work by the customer) to the voltage and current (apparent power) delivered to the load. Power factor should not be confused – as it often is – with another measure of the efficiency of utilization of the power system – *load factor*, which is the relationship of the average power actually used over a period of time (generally measured in kW) to the maximum power used to anytime during the period (also measured in kW).

Power factor is a dimensionless measurement that can range between 0 and 1. A power factor of 1 is called "unity power factor." With unity power factor, all of the voltage and current supplied is converted to either heat or some form of work which may also involve heat (lighting, turning a drive shaft, operating a pump, space heating, etc.) A power factor less than 1 means that some of the voltage and current supplied is temporarily stored as either magnetic energy (inductive load) or as electric energy (capacitive load) and then returned to the system. This process of storing energy either in magnetic coils of inductive motors, for example, or in capacitors causes the power system to be less efficient than it would otherwise be if no energy storage of either form were taking place.

For power to be used by electric consumers it must ultimately be converted to some useable form of work. In converting the energy to work, equipment will often typically be used that converts the electric energy directly to heat, light, etc., or that converts the electric energy to mechanical energy using some type of inductive motor. Consumer loads will therefore consist of resistive loads and inductive loads. Inductive loads (and not capacitive loads) are generally what causes consumer power factors to depart from unity. Almost all consumer loads involve some amount of inductive load. Without getting too far into the underlying mathematics, it is important to understand that in the presence of only resistive load, the oscillations in the current and in the voltage will be perfectly synchronized with one another, as shown in Figure 1. In other words, changes in the polarity of the current signal and of the voltage signal are "in phase" with one another, which means that both signals cross the time axis (the "x-axis") at the same point.

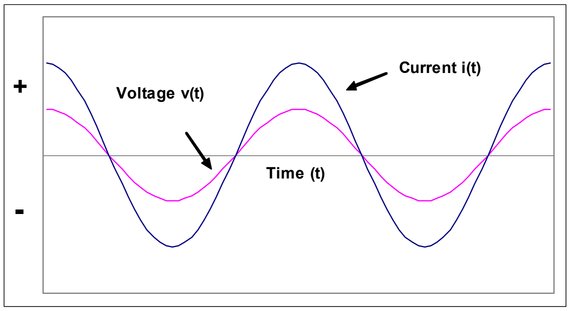

FIGURE 1

With purely resistive load, the instantaneous value of the power, which is equal to the current multiplied by the voltage at any point in time, will always be positive, as shown in Figure 2. Because the power signal never falls below zero, the average power over a full cycle will equal a positive value. With purely resistive load, the load has a unity power factor (a power factor of 1), which means that the current and voltage (apparent power) supplied to the load is equal to the average power consumed by the load.

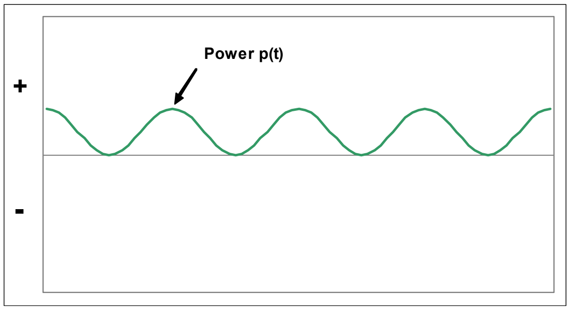

FIGURE 2

With purely inductive load, current lags the voltage by one quarter of a 360° cycle (or by 90°). Because inductive load causes current to lag voltage, the power supplied to any inductive load will oscillate between positive values and negative values, as shown in Figure 3.

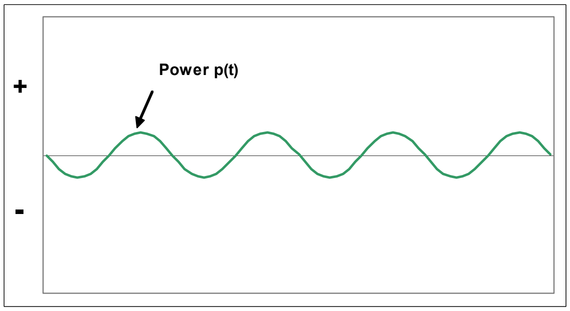

FIGURE 3

What this means is that electric energy is converted and stored as magnetic energy in the inductive coils of a motor (represented by the positive values of power on the graph) and then released before the cycle is finished (represented as the negative values of power on the graph). Because the average power over a full cycle is zero, the power factor of a purely inductive load would be zero. One of the important implications of this is that although electric current and voltage (apparent power) must be supplied to support the energy storage process (for example, the process of storing and releasing magnetic energy in the coils of an inductive motor), none of this stored energy is converted to work. Inductive load therefore utilizes current and voltage supplied without ever being counted or measured as energy used by the customer.

The fact that customers' inductive loads require the use of some of the power system's capacity is ultimately the reason that it is important to measure reactive power and to charge customers for the use of reactive power. A customer's load will typically include both resistive and inductive components. Although the inductive component does not result in any power that would be recorded with standard kWh or kW meters, capacity on the power system is required to deliver the voltage and energy to the inductive component of a customer's load.

In the following sections of this paper, a more mathematical description of these concepts will be developed. For electrical engineers, the remaining sections may serve as a review of the mathematical fundamentals for understanding reactive power and power factor and as an introduction to the tools essential to understanding some of the mathematical algorithms employed in recent reactive power control strategies. For technicians with strong backgrounds in mathematics, but with little formal training in electromagnetism and circuit equations, the remaining sections should provide a useful introduction to this interesting field of study.

III. AC POWER IN THE TIME DOMAIN

AC current and voltage can be described as a time-variant signal in the time domain using the familiar sinusoidal representation. For example, AC current i(t) can naturally be represented as a function of time t as follows:

$$i(t) = I_{\max} \cos(\omega t + \varphi) \qquad (1)$$

Where:
$i(t)$ is the current as a function of time
t is time
I_m is maximum (amplitude) of the current function
ω is the angular frequency of the signal in radians with w=2πf where f is the frequency in hertz
φ is the phase angle in radians at t=0.

Similarly, AC voltage $v(t)$ can be represented as a function of time t as follows:

$$v(t) = V_{\max} \cos(\omega t + \theta) \qquad (2)$$

Where:
$V(t)$ is voltage as a function of time
t is time
V_m is maximum (amplitude) of the current function

ω is the angular frequency of the signal in radians with w=2πf where f is the frequency in hertz

θ is the phase angle in radians at t=0.

AC current i(t) and voltage v(t) (undistorted by higher order harmonics) can be represented graphically in the time domain by the familiar sinusoidal curve:

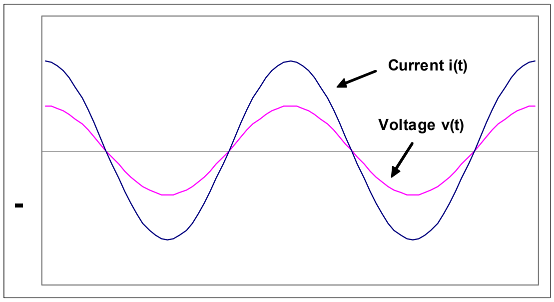

FIGURE 4

In an AC circuit with only resistive load there is no difference between the phase angle φ of the current signal *i(t)* and the phase angle θ of the voltage signal. If the load is purely resistive, then current curve and voltage are perfectly aligned or “in phase” as in the above graph (Figure 4). Since they are in phase, the current and voltage curves cross the time axis (the x-axis) at exactly the same points in time.

This can be illustrated using a portion of a basic single-phase circuit with a resistor representing the resistive load R. The load, for example, could consist of incandescent lighting or a heating coil, both of which have negligible capacitive or inductive components:

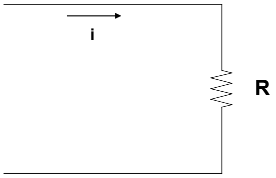

FIGURE 5

According to classical electromagnetic theory, specifically, Ohms’ law, voltage (potential difference) is affected by the resistive load in the circuit as follows:

$$v(t) = i(t)R \tag{3}$$

Of course, this is a fundamental circuit equation in electrical engineering. Resistance is not normally a time-variant operator, but simply scales the amplitude of the current signal; therefore, if i(t)=$I_m$cos(ωt), we have:

$$\begin{aligned} v(t) &= R\,i(t) \\ &= RI_m \cos(\omega t) \\ &= V_m \cos(\omega t) \end{aligned} \tag{4}$$

The power p(t) supplied to the load (also called “instantaneous power” to distinguish it from “average power) is calculated by multiplying the current (at each point in time) by voltage (at each point in time), as follows:

$$\begin{aligned} p(t) &= i(t)\,v(t) \\ &= (I_m \cos(\omega t))(V_m \cos(\omega t)) \\ &= P_m \cos^2(\omega t) \end{aligned} \tag{5}$$

The following graph depicts current i(t), voltage v(t), and power p(t) for a circuit containing only resistive load:

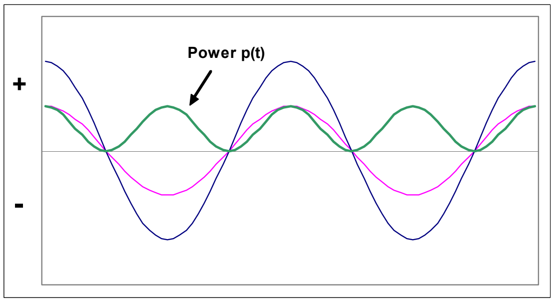

FIGURE 6

A couple of things should be pointed out about these curves. First, it should be noted that while the current and voltage signals reverse polarity over time, going from positive to negative, the power signal p(t) always remains positive, never falling below the time axis (the x-axis). This results from the fact that whenever the current has positive polarity, the voltage also has positive polarity; likewise, whenever the current is negative, the voltage is negative. Consequently, when current is multiplied by voltage at each point in time, the resulting power value (p(t) = i(t)xp(t)) is always positive. This is an extremely important characteristic of purely resistive load, and is not the case in the presence of inductive and capacitive load, as will be discussed shortly. When inductive *or* capacitive loads are present, there will be a difference between the phase angle of the current i(t) and the phase angle of the voltage p(t), and, consequently the power curve p(t) will fall below the time axis. Second, from looking at the mathematics underlying the power curve, we can tell that the average power over one full cycle will likely be equal to one half of the maximum value. This can be verified by integrating the p(t) over 2π radians, as follows:

$$\begin{aligned} P_{avg} &= \frac{1}{2\pi}\int_0^{2\pi} p(t)dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} P_m \cos^2(\omega t)dt \\ &= \frac{P_m}{2\pi}\left[\frac{1}{2}t + \frac{1}{4\omega t}\sin(2\omega t)\right]_0^{2\pi} \qquad (6) \\ &= \frac{P_m}{2\pi}\left[\frac{1}{2}2\pi\right] \\ &= \frac{P_m}{2} \end{aligned}$$

In this and subsequent examples, it is assumed that the phase angles at the source are zero, thus simplifying the mathematics somewhat and reducing the clutter in the expressions. Since we can always choose the integration boundaries in such a way that the phase differences disappear, this assumption does not yield results lacking in generality.

The presence of inductive loads in a circuit alters the phase angle of the voltage. Consider a portion of a circuit with inductive load L, as illustrated below:

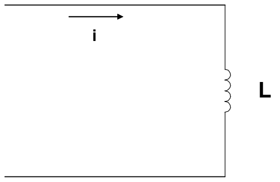

FIGURE 7

As a result of Faraday's law of induction, voltage is affected as follows from the presence of inductive load in the circuit:

$$v(t) = L\frac{d}{dt}i(t) \qquad (7)$$

Now, if i(t) = I_m cos (ωt), we have:

$$\begin{aligned} v(t) &= L\frac{d}{dt}(I_m \cos(\omega t)) \\ &= -LI_m \sin(\omega t) \qquad (8) \\ &= V_m \cos(\omega t + \pi/2) \end{aligned}$$

In the presence of only inductive load, the voltage has now become a negative sine function, which is 90° (π/2 radians) out of phase with the current.

Thus, inductive load without any resistive load (a theoretical construct which would not occur in practice) results in a 90° difference between the phase angle of the current and the phase angle of the voltage. In this situation, the current is said to "lag" the voltage. In a circuit with only inductive load, instantaneous power p(t) becomes:

$$\begin{aligned} p(t) &= i(t)v(t) \\ &= (I_m \cos(\omega t))(V_m \cos(\omega t + \pi/2)) \qquad (9) \\ &= P_m \cos(\omega t)\cos(\omega t + \pi/2) \end{aligned}$$

The following graph shows the power p(t), current i(t), and voltage v(t) for a single-phase circuit consisting of only inductive load:

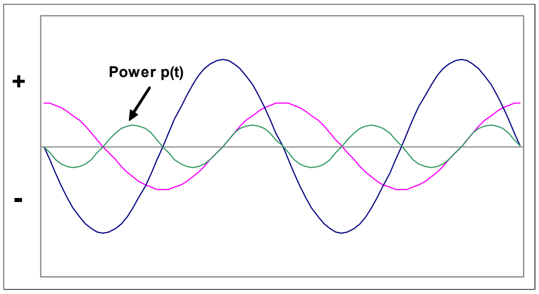

FIGURE 8

It should be noted that the power curve p(t) now oscillates between a maximum positive value and a maximum negative value, alternating between negative values and positive values over time. This is in stark contrast to the case of purely resistive load in which instantaneous power p(t) always remains positive. Whenever current has the opposite sign from the voltage, power p(t) at that time is negative,; but whenever, current and voltage have the same signs, either positive or negative, power p(t) is positive.

It should also be noted that the average power over a full cycle (or any period of n2π radians) is zero. This characteristic has enormous implications, underscoring the significance of reactive power. What this shows is that inductive load in a circuit does not consume <u>any</u> power over a full cycle. (We are referring here to a full cycle of the power, which, it should be noted, oscillates at *twice* the rate of current and voltage and thus occurs in *half* the time.) During a cycle, electric energy (or power) is supplied to an inductor and temporarily converted into magnetic energy in the inductor coils. But before the cycle is completed, the magnetic energy is released from the inductor and converted back into electric energy, ultimately resulting in a zero use of energy (ignoring losses).

It is important to realize that even though an inductor does not ultimately use any power to perform work, and ultimately does not remove any energy from the system (other than losses), the system must have sufficient capacity to provide the maximum instantaneous power P′ supplied to the inductive load. This is critically important point for understanding the impact of inductive load on a utility system – sufficient supply capacity and transmission and distribution capacity must be available to supply the inductive load. In other words, inductive load takes up capacity on the system. It should be readily apparent that the process of storing (and releasing) magnetic energy in (and from) an inductor must "consume" capacity in a power system. Furthermore, supplying inductive load involves power losses. Specifically, the cycling of power to serve inductive load results in line losses.

To demonstrate that the average power over one full cycle is zero for a circuit consisting of only inductive load, we need to integrate the power signal over a period of 2π radians and divide by the same period, as follows:

$$\begin{aligned} P_{avg} &= \frac{1}{2\pi}\int_0^{2\pi} p(t)dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} P_m \cos(\omega t)\sin(\omega t)dt \\ &= \frac{P_m}{2\pi}\left[\frac{1}{w}\sin^2(\omega t)\right]_0^{2\pi} \qquad (10) \\ &= \frac{P_m}{2\pi}\cdot 0 \\ &= 0 \end{aligned}$$

The presence of capacitive load in a circuit also creates a difference between the phase angle of current i(t) and the phase angle of voltage v(t), but in exactly the opposite direction as inductive load. It is for this reason that capacitors can be used to compensate for inductive load on a system and that inductors (or reactors) can be used to compensate for capacitive load on a system. Specifically, the presence of capacitive load affects the phase angle of the current. Consider a portion of a circuit with only capacitive load, as illustrated by the following circuit segment:

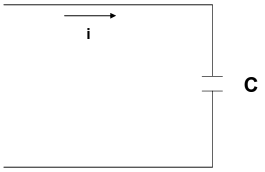

FIGURE 9

Much as an inductor acts as a differential operator on current, a capacitor acts as a differential operator on voltage. The effect of capacitive load on the current is determined by the following:

$$i(t) = C\frac{d}{dt}v(t) \qquad (11)$$

Now, if $v(t) = V_m \cos(\omega t)$, we have:

$$\begin{aligned} i(t) &= C\frac{d}{dt}(V_m \cos(\omega t)) \\ &= -CV_m \sin(\omega t) \qquad (12) \\ &= I_m \cos(\omega t + \pi/2) \end{aligned}$$

In the presence of capacitive load, the current has now become a negative sine function, which again is 90° (π/2 radians) out of phase with the voltage. Thus, the current is now said to "lead" the voltage. The fact that a capacitor has the effect of shifting the phase angle current (or voltage) in the opposite direction from a inductor is the reason that capacitors are used to compensate for inductive load on a system. As with inductive load, the average power consumed by a capacitor (ignoring losses) is zero.

It was stated earlier that an inductor converts electric energy to *magnetic energy*, storing and releasing the energy throughout a cycle. Capacitors store and release *electric energy* throughout a cycle. In the absence of inductive load, capacitors (or equipment acting as capacitors) would have the same diminishing effect on capacity as inductive load. But because inductive load is so ubiquitous in industrial, commercial and residential power applications, capacitive load often represents less of a problem than inductive load, especially during peak load conductions. During low-load conditions, however, capacitive load can be problematic.

IV. PHASE SHIFTS IN AN RL CIRCUIT

Thus far we have considered examples that included either resistive load, inductive load, or capacitive load, but not a combination of the three elements. Reactive power results from a combination of resistive load with either inductive load or capacitive load. The relative relationships between resistive load and either inductive or capacitive load will determine the shift in the phase angle of the current or voltage and will ultimately determine the amount of reactive power consumed or provided. In the previous examples, we considered partial circuits with fixed currents flowing through circuit elements. In the following example, we will develop a detailed result by solving a differential equation of a complete circuit consisting of a resistor and inductor. It is assumed that at t=0 the RL circuit is closed:

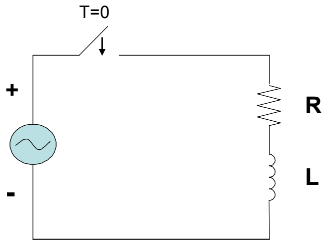

FIGURE 10

If the source voltage is $V_s(t) = V_m\cos(\omega t)$, then applying Kirchhoff's Voltage Law (KVL) (which states that the sum of the voltages for a circuit is zero) for the circuits implies that:

$$L\frac{di}{dt} + Ri = V_m \cos(\omega t) \qquad (13)$$

This right hand side of this non-homogenous differential equation is of a form that allows us to solve it using the method of undetermined coefficients.[i] Recall that a general solution to this type of differential equation consists of a *complementary function* i_c and a *particular solution* i_p. The particular solution is of the form:

$$i_p = A\cos(\omega t) + B\sin(\omega t) \qquad (14)$$

Taking the derivative gives:

$$\frac{di_p}{dt} = -\omega A\cos(\omega t) + \omega B\cos(\omega t) \qquad (15)$$

Substituting (14) and (15) into the differential equation (13) gives:

$$\begin{aligned} &L(\omega B\cos\omega t - \omega A\sin\omega t) + \\ &\quad R(A\cos\omega t + B\sin\omega t) = V_m \cos(\omega t) \end{aligned} \qquad (16)$$

By equating the terms on the right side of (16) with those on the left produces the following system of equations:

$$\begin{aligned} RA + \omega LB &= V_m \\ -\omega LA + RB &= 0 \end{aligned} \qquad (17)$$

Solving the two equations simultaneously results in the following values for A and B necessary to solve the particular solution i_p:

$$A = \frac{RV_m}{\omega^2 L^2 + R^2} \qquad B = \frac{\omega L V_m}{\omega^2 L^2 + R^2} \tag{18}$$

The complementary function can be determined by solving the differential equation as a homogeneous equation, as follows:

$$L\frac{di}{dt} + Ri = 0 \qquad i_c = ke^{\frac{-R}{L}t} \tag{19}$$

The general solution to the differential equation can be derived by adding the particular solution and the complimentary function, as follows:

$$\begin{aligned} i &= i_p + i_c \\ &= ke^{\frac{-R}{L}t} + \frac{RV_m}{\omega^2 L^2 + R^2}\cos(\omega t) \\ &\quad + \frac{\omega L V_m}{\omega^2 L^2 + R^2}\sin(\omega t) \end{aligned} \tag{20}$$

At t=0 there is not current; therefore,

$$k = \frac{-RV_m}{\omega^2 L^2 + R^2} \tag{21}$$

thus yielding the following solution to the differential equation:

$$\begin{aligned} i = \frac{-RV_m}{\omega^2 L^2 + R^2}e^{\frac{-R}{L}t} + \frac{RV_m}{\omega^2 L^2 + R^2}\cos(\omega t) \\ + \frac{\omega L V_m}{\omega^2 L^2 + R^2}\sin(\omega t) \end{aligned} \tag{22}$$

The first term in (22) is a *transient term* which will die off over time. The two other terms represent the steady-state current. Since we are concerned with steady-state conditions and not transient conditions, the transient term can be dropped, leaving:

$$\begin{aligned} i = \frac{RV_m}{\omega^2 L^2 + R^2}\cos(\omega t) \\ + \frac{\omega L V_m}{\omega^2 L^2 + R^2}\sin(\omega t) \end{aligned} \tag{23}$$

or

$$\begin{aligned} i = \frac{V_m}{\sqrt{\omega^2 L^2 + R^2}}\Bigg(\frac{R}{\sqrt{\omega^2 L^2 + R^2}}\cos(\omega t) \\ + \frac{\omega L}{\sqrt{\omega^2 L^2 + R^2}}\sin(\omega t) \Bigg) \end{aligned} \tag{24}$$

The Pythagorean relationship is evident in the two terms in the brackets. We can therefore restate the first term's coefficient as a sine function of an angle φ and the second term's coefficient as a cosine of the angle φ, as follows:

$$\begin{aligned} i = \frac{V_m}{\sqrt{\omega^2 L^2 + R^2}}(\sin(\varphi)\cos(\omega t) \\ + \cos(\varphi)\sin(\omega t)) \end{aligned} \tag{25}$$

From the angle subtraction formula in basic trigonometry, we know that:

$$\begin{aligned} \sin(\varphi)\cos(\omega t) + \cos(\varphi)\sin(\omega t)) \\ = \cos(\omega t + \varphi) \end{aligned} \tag{26}$$

Therefore, we have two equivalent representations of the current i(t), one representing current in terms of a phase shift φ and the other representing current in terms of the relative, or "weighted", relationship between the amount of reactance and inductance in the circuit:

$$\begin{aligned} i(t) &= \frac{V_m}{\sqrt{\omega^2 L^2 + R^2}}\cos(\omega t + \varphi) \\ &= \frac{RV_m}{\omega^2 L^2 + R^2}\cos(\omega t) + \frac{\omega L V_m}{\omega^2 L^2 + R^2}\sin(\omega t) \end{aligned} \qquad (27)$$

What this tells us it that the phase shift φ is determined entirely by the relative relationship between the resistive load and the inductive load in the circuit.

An important attribute of the last equation from a mathematical perspective is that current is constructed as a linear combination of two orthogonal functions. The concept of *orthogonal functions* is a generalization of perpendicular lines or vectors in Euclidean space, as in the x- and y-axes of a graph. Two functions are orthogonal if their scalar product is 0, which is analogous to the scalar product of the two unit basis vectors $\hat{\mathbf{\imath}}$ and $\hat{\mathbf{\jmath}}$ being zero. The fact that current (and voltage) can be constructed as a linear combination of the sine and cosine functions may immediately bring to mind that practically any periodic function can be constructed using a Fourier series by representing the function as a linear combination of an infinite series of orthogonal functions. In fact, equation (25) is equivalent to a Fourier series representation of current in the fundamental frequency term (i.e., the first order harmonic of a trigonometric Fourier series.) In the absence of higher order harmonics, current consists of only terms in cos(nωt) and sin(nωt)), with n = 1. In practice, higher order harmonics are always present to some extent. Current, voltage and power can thus be represented in terms of the trigonometric Fourier series:

$$f(t) = \frac{\alpha_0}{2} + \sum_{i=1}^{\infty} \alpha_i \cos(i\omega t) + \beta_i \sin(i\omega t) \qquad (28)$$

The zeroth order term ($\alpha_0/2$) vanishes for current and voltage. For power p(t), the zeroth order term is equal to average power and is referred to as the “dc component” of the power.

Equation (25) tells us that in the absence of higher order harmonics, *any* sinusoidal current in the fundamental frequency term can be stated as a linear combination of coefficients in cos(ωt) and coefficients in sin(ωt). The orthogonal functions cos(ωt) and sin(ωt) are said to form a *basis* of *all* first order sinusoidal functions. As a result, we can perform equivalent calculations in other domains (for example, the phasor domain) by identifying alternative basis functions or basis vectors that do not depend on the time variable, which greatly simplifies calculations, specifically calculations involving derivatives.[ii]

V. REACTIVE POWER AND POWER FACTOR

From our discussion thus far it should be clear that capacity is required to serve inductive and capacitive load. As shown in Figure 5, even though the average power over a full cycle is zero, the power supply system would still need to be able to provide a maximum instantaneous power of $p_m(t)$ to serve a purely inductive load. The same is true of a purely capacitive load. The concept of reactive was developed to account for the power necessary to serve inductive and capacitive component of the load.

Before defining reactive power and power factor, it is necessary to describe how a *scalar product* and a L^2 *norm,* or *root mean square* (RMS), of functions in $L^2(0,2\pi)$ space[iii] are calculated. A *scalar product* +f(t),g(t), of two functions f(t), g(t) in $L^2(0,2\pi)$ is calculated as follows:

$$\langle f(t), g(t)\rangle = \frac{1}{2\pi}\int_o^{2\pi} f(t)g(t)dt \qquad (29)$$

The L^2 *norm* of a function, (2f(t)2 or F_{rms}) is calculated for any function f(t) in $L^2(0,2\pi)$ as follows:

$$\|f(t)\| = F_{rms} = \sqrt{\frac{1}{2\pi}\int_o^{2\pi} f^2 dt} \qquad (30)$$

In mathematical analysis, the Schwarz inequality tells us that:

$$\langle f(t), g(t)\rangle \le \|f(t)\|\,\|g(t)\| \qquad (31)$$

From the Schwarz inequality it follows that there must exist an angle φ between 0 and π such that:

$$\cos(\varphi) = \frac{\langle f(t), g(t)\rangle}{\|f(t)\| \, \|g(t)\|} = \frac{\frac{1}{2\pi}\int_o^{2\pi} f(t)g(t)dt}{\sqrt{\frac{1}{2\pi}\int_o^{2\pi} f^2 dt}\sqrt{\frac{1}{2\pi}\int_o^{2\pi} g^2 dt}} \tag{32}$$

This is a general result that applies to virtually any square integrable function, including functional representations of current and energy.[iv] Replacing f(t) with current i(t) and replacing g(t) with voltage v(t) in equation (32), we get:

$$\cos(\varphi) = \frac{\langle i(t), v(t)\rangle}{\|i(t)\| \, \|v(t)\|} = \frac{\frac{1}{2\pi}\int_o^{2\pi} i(t)v(t)dt}{\sqrt{\frac{1}{2\pi}\int_o^{2\pi} i(t)^2 dt}\sqrt{\frac{1}{2\pi}\int_o^{2\pi} v(t)^2 dt}} = \frac{P_{avg}}{I_{rms}V_{rms}} \tag{33}$$

Cos(φ) is called the "power factor" (or "PF") and is determined by the phase difference between the current and the voltage. This is a very general result, which will apply for vectors in l^2 space as well as for functions in L^2 space. We can either use a functional representation of voltage and current and perform the calculations in terms of P_{avg}, I_{rms}, and V_{rms}, or perform the calculations in terms of vector (or phasor/complex power) representations of I, V, and P.

The formula also applies for periodic functions that include higher order harmonics. However, we need to be careful about the meaning of the power factor angle *φ* in the presence of harmonics. While $P_{avg} \div I_{rms}V_{rms}$ represents the "true" power factor, the power factor angle does not represent the phase angle of the fundamental frequency; but rather, it represents the phase angle resulting from all of the harmonics in the current and voltage signals. Current i(t) and voltage v(t) can be viewed as generalized vectors in an infinite-dimensional (or n-dimensional) space rather than in 2 dimensions suggested by a basic sinusoidal representation of current and voltage, as in equation (27). For example, in the presence of steady-state harmonics, voltage and current can be represented in terms of a Fourier series, as follows:

$$\begin{aligned} i(t) &= \sum_{i=1}^{\infty} I_i^R \cos(i\omega t) + I_i^Q \sin(i\omega t) \\ &= \sum_{i=1}^{\infty} I_i \cos(i\omega t + \delta_i) \\ v(t) &= \sum_{i=1}^{\infty} V_i^R \cos(i\omega t) + V_i^Q \sin(i\omega t) \\ &= \sum_{i=1}^{\infty} V_i \cos(i\omega t + \theta_i) \end{aligned} \tag{34}$$

Where I_i^R and V_i^R are the coefficients representing the i^th order real (or resistive) components of the signals; I_i^Q and V_i^Q are the coefficients representing the i^th order reactive (capacitive and/or inductive) components of the signals; and δ_i and θ_i are the phase shifts for the i^th order term. These coefficients can be viewed as components of two generalized vectors with infinite dimension. The power factor angle *φ* in equation (33) therefore represents the angle between the generalized vector i(t) and the generalized vector v(t). Again, the power factor angle *φ does not* represent the phase angle associated with the fundamental frequency (i.e., the 1st order term where i=1 in the Fourier series) in the presence of harmonics. But the power factor angle *φ does* represent the total or effective phase angle. In this sense, the angle *φ* can be considered to be the "true" power factor angle of the current and voltage.[v]

Equation (33) implies that there exists a quantity or a function that is in quadrature with P_{avg} (or separated by a phase of π/2 or 90° with P_{avg}). This quantity which will be labeled Q is referred to as "reactive power". If we define "apparent power" S as $S=I_{rms}V_{rms}$ and "real power" P as P_{avg} then we have the following relationships:

$$\begin{aligned} PF &= \frac{P}{S} \\ S^2 &= P^2 + Q^2 \\ S &= \sqrt{P^2 + Q^2} \end{aligned} \tag{35}$$

with real power P measured in KW, apparent power S measured in KVA, and reactive power Q measured in KVAR. Power Factor PF is a dimensionless number that can range between 0 and 1. Consequently, real power P can never exceed S and PF can never exceed 1. Reactive Power Q can be either negative or positive, depending on the direction of the phase shift between current and voltage. It is positive if the power factor is lagging (with inductive load exceeding the capacitive load) and it is negative if the power factor is leading (with the capacitive load exceeding the inductive load), although the sign is arbitrary.

VI. CONCLUSION

Much of the current research into reactive power control relies on the mathematical formalism described in this paper. Particularly interesting research from this writer's perspective involves developing control algorithms using vector and matrix concepts derived from mathematical analysis. For example, vector and Hilbert space decomposition techniques have recently been investigated for use as computational tools for analyzing compensation strategies for improving voltage response of power systems with non-sinusoidal signals.[vi] It is also the author's view that acquiring a better understanding of the underlying mathematics involved in reactive power and power factor calculations would help utility rate, engineering and marketing technicians provide better explanations of these concepts to regulators and utility customers.

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[i] The method of undetermined coefficients is covered in just about any recent differential equations text. For example, see Shepley L. Ross, *Introduction to Ordinary Differential Equations* 4th Edition (John Wiley & Sons, 1989.)

ii One of the most useful texts on reactive power control, T. J. E. Miller, *Reactive Power Control Systems* (Wiley & Sons, 1982), discusses reactive power almost exclusively in terms of phasors and complex power.

[iii] A function is an element of $L^2(0,2\pi)$ if it is a measurable, square integrable, real valued function on the real interval $[0,2\pi]$. L^2 spaces are discussed in Walter Rudin, *Principles of Mathematical Analysis* 3d Edition (McGraw-Hill, 1976) and Richard L. Wheeden and Antoni Zygmund, *Measure and Integral: An Introduction to Real Analysis* (Marcel Dekker, Inc., 1977). The $L^2(0,2\pi)$ space is covered in Robert S. Borden, *A Course in Advanced Calculus* (Elsevier Science Publishing Co., Inc, 1983.)

[iv] See G. Sansone, *Orthogonal Functions* (Dover Publications, Inc, 1991), p. 6.

[v] See W. Mack Grady and Robert J. Gilleskie, "Harmonics and how they relate to power factor," *Proc. Of the EPRI Power Quality Issues & Opportunities Conference* (PQA '93), San Diego, CA, November 1993, and George J. Wakileh, *Power System Harmonics: Fundamentals, Analysis and Filter Design* (Springer, 2001). For a description of generalized vectors and their relationship to Fourier series see George P. Tolstov, *Fourier Series* (Prentice Hall, 1963).

[vi] For example, Hanoch Lev Ari and Aleksandar M. Stankovic, "Hilbert Space Techniques for Modeling and Compensation of Reactive Power in Energy Processing Systems," *IEEE Transactions on Circuits and Systems – I Fundamental Theory and Applications*, Vol. 50, No. 4, April 2003; Niels LaWhite and Marija D. Ilić, "Vector Space Decomposition of Reactive Power for Periodci Nonsinusoidal Signals," ," *IEEE Transactions on Circuits and Systems – I Fundamental Theory and Applications*, Vol. 44, No. 4, April 1997.